PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

# Electromagnetic Integrals

G. Greenhill

*Phil. Trans. R. Soc. Lond. A* 1920 **220**, 35-65
doi: 10.1098/rsta.1920.0002

# II. *Electromagnetic Integrals.*

*By Sir* G. Greenhill, *F.R.S.*

(Received April 22,—Read May 9, 1918.)

These are the integrals, elliptic integrals (E.I.) for the most part, and of the first, second and third kind (I., II., III., E.I.) arising in the practical problem of the measurement and determination of the electrical units, for their regulation and definition by Act of Parliament in commercial use.

The experiments have been carried out with the ampère balance invented by Viriamu Jones, also with the Lorenz apparatus for measuring resistance ('Phil. Mag.,' 1889, 'Phil. Trans.,' 1891, 1913), constructed at the charge of Sir Andrew Noble and the British Association, in use at the National Physical Laboratory (N.P.L.), Teddington.

A description of this current weigher, ampère balance, is given by Ayrton, Mather and F. E. Smith in the 'Phil. Trans.,' A, vol. 207, 1907, and the theory is developed, with a description of the accuracy of measurement obtainable, to be recorded in the Act of Parliament. Also of the Lorenz apparatus, by F. E. Smith, in 'Phil. Trans.,' 1913.

Our object here is to revise and simplify the mathematical treatment required in these calculations and to present the theory in a form adapted for elementary instruction; at the same time, to reconcile the notation and results of the various writers, Viriamu Jones, G. M. Minchin, and others, and to standardise them in accordance with Maxwell's 'Electricity and Magnetism' (E. and M.).

1. Starting then on § 703, E. and M., it can be shown that all the results required can be made to originate and grow out of Maxwell's expression for M, mutual induction of two parallel circular currents on the same axis, in circles of radius OB = $a$ and NP = A, a distance ON = $b$ apart, as on fig. 1 and 2 (p. 38), and M is given in his notation by

$$(1) \qquad M = \int_0^{2\pi} 2\pi A a \cos\theta \frac{d\theta}{PQ}, \qquad PQ^2 = A^2+2Aa\cos\theta+a^2+b^2, \qquad AOQ = \theta,$$

and M is expressible by the complete E.I., I. and II.

 [Published December 15, 1919.

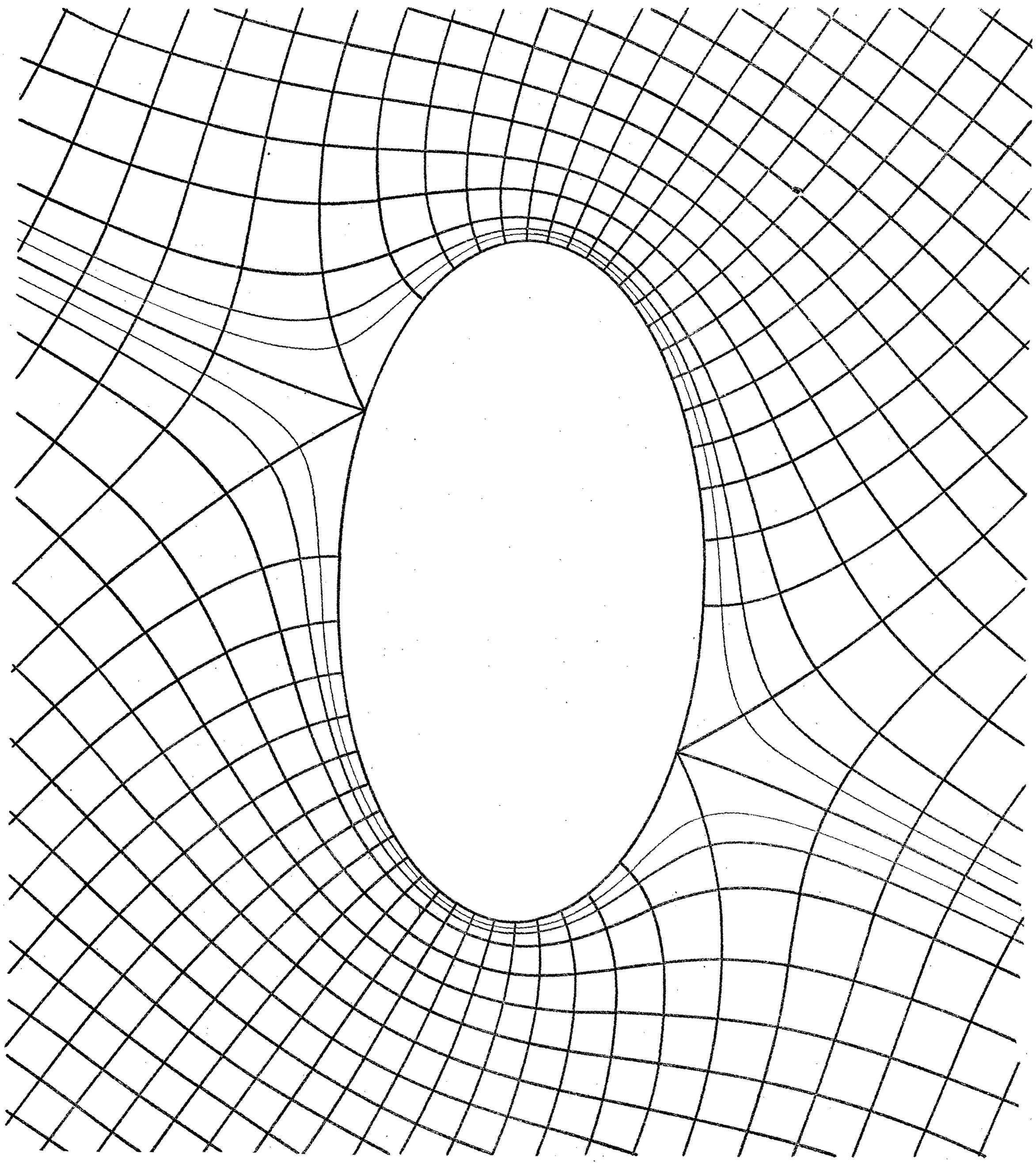

Fig. 3.

A re-drawing has been made and a lantern slide, of MAXWELL'S fig. XVIII, of the curves of constant M, or lines of magnetic force of the circular current round the circle AB perpendicular to the plane, on the diameter joining the foci, in which the confocal co-ordinates were employed of WEIR'S azimuth diagram.

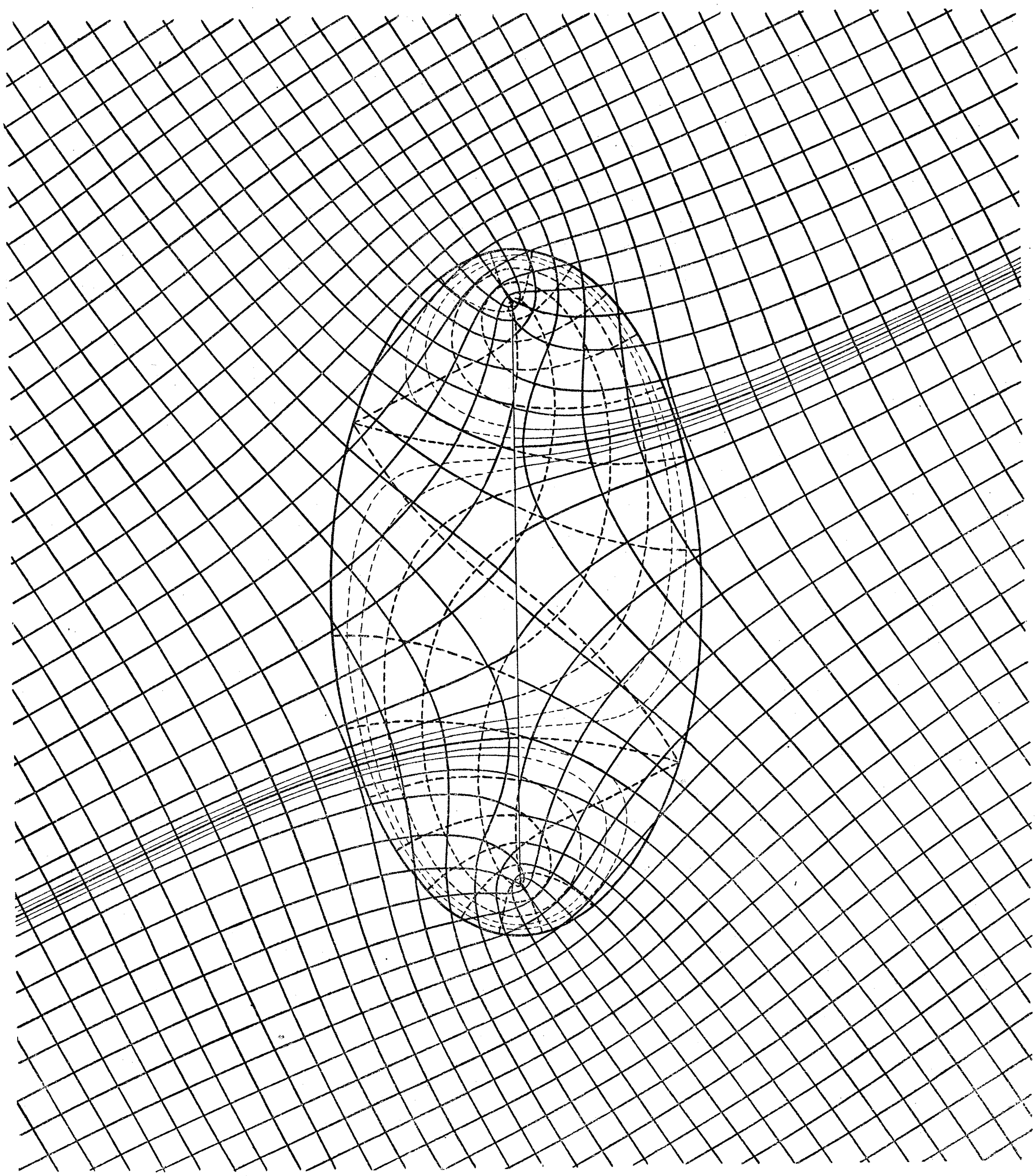

Fig. 4.

This was carried out by Mr. J. W. Hicks for comparison with Maxwell's figure using the formulas given later in § 12.

The same confocal co-ordinates on the Weir chart were employed by Colonel Hippisley in drawing fig. 3 and 4, and the lantern slides, showing the lines of uniplanar flow past an ellipse; but here the co-ordinate ruling has been rubbed out.

The conformal representation of the mapping connecting $z = x+iy$ and $\zeta = \eta+i\xi$ is given by $z = c \operatorname{ch} \zeta$; and then with $w = \phi+i\psi$, $\gamma = \beta+i\alpha$, the relation $w = \operatorname{ch}(\zeta-\gamma)$ gives the mapping of $\phi$, $\psi$, the velocity and stream function of the motion, so that

$$(2) \qquad \phi = \operatorname{ch}(\eta-\alpha)\cos(\xi-\beta), \qquad \psi = \operatorname{sh}(\eta-\alpha)\sin(\xi-\beta).$$

Fig. 3 gives the ordinary stream motion of the distortion of a uniform current by an elliptic post, and should figure as the typical diagram in a treatise on Hydrodynamics. But fig. 4 is curious in showing the analytical prolongation of the functions

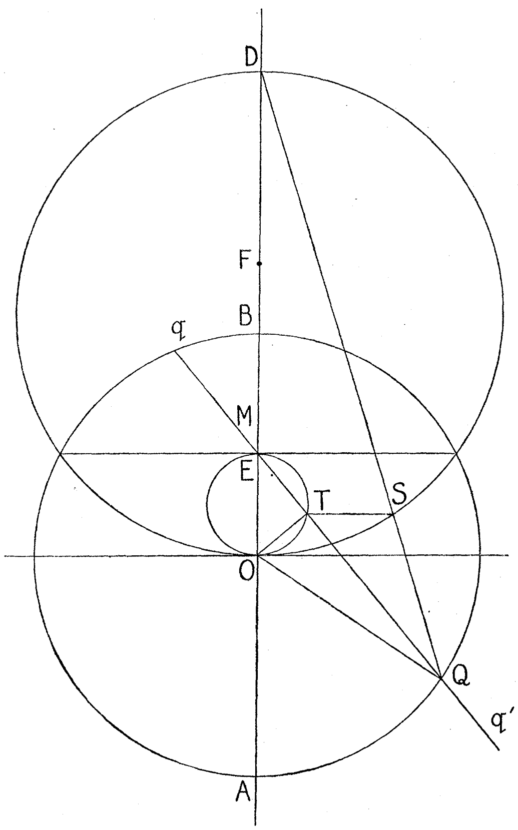


Fig. 1.

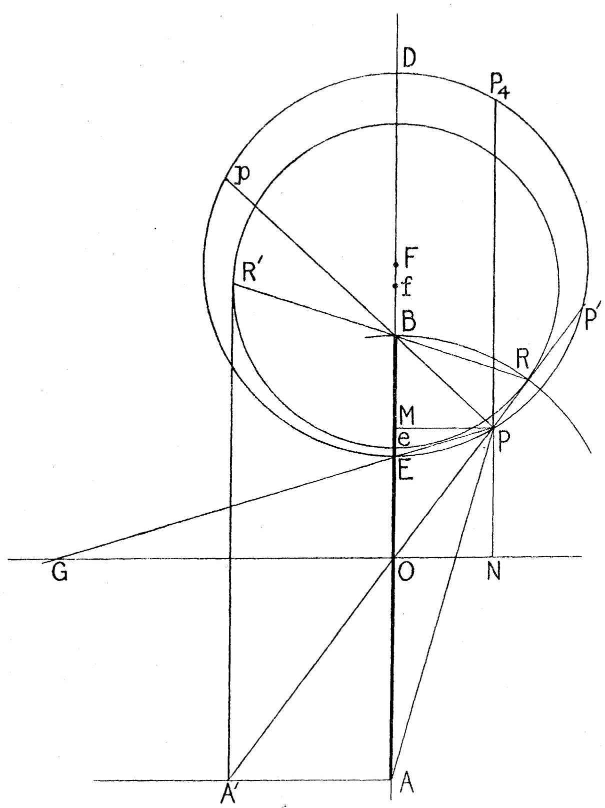


Fig. 2.

for $\eta < \alpha$ on the Riemann sheet, with the cut along the line SS′ joining the foci. It shows the middle stream coming to a waterfall across SS′ and circulating in a whirlpool chamber in the interior of the ellipse $\eta = \alpha$, and then emerging in another stream off to infinity.

The Weir chart would be ideal to employ in plotting some of the curves described by LEGENDRE (F.E., I., p. 411), orbits invented by EULER, 1760, under two centres of gravitation at S, S′, as the variables employed by EULER and LEGENDRE are $\tan \frac{1}{2}\xi$, $\operatorname{th} \frac{1}{2}\eta$, and these are separated in the equations of motion.

2. An integration of M with respect to $b$ will lead immediately to V. JONES's expression for the mutual induction, L, between the circular current PP′ on the diameter 2A, and a uniform current sheet flowing round the cylinder on the diameter $2a$, stretching from the circle AB a length $b$, up to the circle PP′; drawing out the circle AQB axially, like a concertina.

The current sheet is taken as the equivalent of the close helical winding in the ampère balance of the wire on the cylinder carrying the electric current and forming a solenoidal magnet, of which a constant L gives a line of magnetic force, the one passing through P, these lines circulating through the solenoidal tube and closing again outside.

In the hydrodynamical analogue L would be the stream function (S.F.) of liquid circulating through the tube.

Employing the lemma of the integral calculus, for the line potential of MP at Q,

$$(1)\qquad \int\frac{db}{\mathrm{PQ}} = \mathrm{th}^{-1}\frac{b}{\mathrm{PQ}} = \mathrm{sh}^{-1}\frac{b}{\mathrm{MQ}} = \mathrm{ch}^{-1}\frac{\mathrm{PQ}}{\mathrm{MQ}}, \qquad \mathrm{MQ}^2 = \mathrm{A}^2+2\mathrm{A}a\cos\theta+a^2,$$

$$(2)\qquad \mathrm{L} = \int_0^b \mathrm{M}\,db = \int_0^b\int_0^{2\pi} 2\pi\mathrm{A}a\cos\theta\,\frac{d\theta\,db}{\mathrm{PQ}} = \int_0^{2\pi} 2\pi\mathrm{A}a\cos\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}\,d\theta,$$

and integrating by parts, with the lemma of the differential calculus

$$(3)\qquad \frac{d}{d\theta}\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}} = \frac{\mathrm{A}a\sin\theta}{\mathrm{MQ}^2}\cdot\frac{b}{\mathrm{PQ}},$$

$$(4)\qquad \mathrm{L} = 2\pi\mathrm{A}a\left(\sin\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}\right)_0^{2\pi} - \int 2\pi\mathrm{A}a\sin\theta\,\frac{\mathrm{A}a\sin\theta}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}}$$

$$= * - \int 2\pi\,\frac{\mathrm{A}^2a^2\sin^2\theta}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}},$$

the $*$ marking a term which vanishes at the limits, and with

$$(5)\qquad 4\mathrm{A}^2a^2\sin^2\theta = 4\mathrm{A}^2a^2-(\mathrm{MQ}^2-a^2-\mathrm{A}^2)^2$$

$$= -\mathrm{MQ}^2+2(a^2+\mathrm{A}^2)\,\mathrm{MQ}^2-(a^2\mathrm{A}^2)^2,$$

$$(6)\qquad \mathrm{L} = \tfrac{1}{2}\pi\int\left[-\mathrm{MQ}^2+2(a^2+\mathrm{A}^2)-\frac{(a^2-\mathrm{A}^2)^2}{\mathrm{MQ}^2}\right]\frac{b\,d\theta}{\mathrm{PQ}}$$

$$= \tfrac{1}{2}\pi\int\left[a^2+\mathrm{A}^2-2\mathrm{A}a\cos\theta-\frac{(a^2-\mathrm{A}^2)^2}{\mathrm{MQ}^2}\right]\frac{b\,d\theta}{\mathrm{PQ}},$$

introducing the complete elliptic integral, I., II., III.; and this is the expression employed by V. JONES, but obtained by a complicated dissection.

3. In the reduction of these integrals to a standard form, for the purpose of a numerical calculation by use of the tables of LEGENDRE, it is convenient to put $\theta = 2\omega$, $\omega = \text{ABQ}$, in fig. 1, and to introduce a new variable $t$, and constants $\tau$, $t_1$, $t_2$, $t_3$, in accordance with the notation of WEIERSTRASS, such that, $m$ denoting a homogeneity factor,

(1) $$\text{PQ}^2 = r^2 = m^2(t_1-t), \qquad \text{MQ}^2 = r^2-b^2 = m^2(\tau-t), \qquad \text{PM}^2 = b^2 = m^2(t_1-\tau),$$
$$\text{PA}^2 = r_3^2 = m^2(t_1-t_3), \qquad \text{PB}^2 = r_2^2 = m^2(t_1-t_2),$$
$$\text{MA}^2 = r_3^2-b^2 = (a+\text{A})^2 = m^2(\tau-t_3), \qquad \text{MB}^2 = r_2^2-b^2 = (a-\text{A})^2 = m^2(\tau-t_2),$$
$$\text{PA}^2-\text{PQ}^2 = r_3^2-r^2 = 2\text{A}a(1-\cos\theta) = 4\text{A}a\sin^2\omega = m^2(t-t_3),$$
$$\text{PQ}^2-\text{PB}^2 = r^2-r_2^2 = 2\text{A}a(1+\cos\theta) = 4\text{A}a\cos^2\omega = m^2(t_2-t),$$
$$\text{PA}^2-\text{PB}^2 = r_3^2-r_2^2 = 4\text{A}a = m^2(t_2-t_3),$$

(2) $$\infty > t_1 > \tau > t_2 > t > t_3 > \infty.$$

In the notation of LEGENDRE

(3) $$\text{PQ}^2 = r^2 = r_3^2\cos^2\omega + r_2^2\sin^2\omega = r_3^2\,\Delta^2(\omega,\gamma), \qquad \gamma' = \frac{r_2}{r_3},$$

(4) $$\frac{d\theta}{\text{PQ}} = \frac{2d\omega}{r_3\,\Delta\omega}, \qquad \text{P} = \int_0^{2\pi}\frac{a\,d\theta}{\text{PQ}} = \frac{2a}{r_3}\int_0^{\pi}\frac{d\omega}{\Delta\omega} = \frac{4a\text{G}}{r_3},$$

and P is the rim potential of the circle on AB, with

(5) $$\text{G} = \int_0^{\frac{1}{2}\pi}\frac{d\omega}{\Delta(\omega,\gamma)}, \qquad \text{P} = \frac{4a}{\sqrt{(r_2r_3)}}\,\text{G}\sqrt{\gamma'},$$

(6) $$\text{Q} = \int_0^{2\pi}\frac{-a\cos\theta\,d\theta}{\text{PQ}} = \int_0^{\pi}(2\sin^2\omega-1)\frac{2a\,d\omega}{r_3\,\Delta\omega}$$
$$= \frac{4a}{\gamma^2 r_3}\int_0^{\frac{1}{2}\pi}[2(1-\Delta^2\omega)-\gamma^2]\frac{d\omega}{\Delta\omega}$$
$$= \frac{4a}{\gamma^2 r_3}[(2-\gamma^2)\text{G}-2\text{H}], \qquad \text{H} = \text{E}(\gamma) = \int_0^{\frac{1}{2}\pi}\Delta(\omega,\gamma)\,d\omega,$$

(7) $$\text{M} = -2\pi\text{QA} = -2\pi r_3[(2-\gamma^2)\text{G}-2\text{H}],$$

and $\text{Q}\cos\phi$ is the magnetic potential of the circle on AB, with uniform magnetisation parallel to AB.

In the Third Elliptic Integral keep to the Weierstrassian form, with the variable $t$,

(8) $$m^2\,dt = 2\text{A}a\sin\theta\,d\theta = m^2\sqrt{(t_2-t\,.\,t-t_3)}\,d\theta,$$

(9) $$\frac{d\theta}{\text{PQ}} = \frac{2dt}{m\sqrt{\text{T}}}, \qquad \frac{b\,d\theta}{\text{PQ}} = \frac{2\sqrt{(t_1-\tau)}\,dt}{\sqrt{\text{T}}},$$

(10) $$T = 4 \,.\, t_1-t \,.\, t_2-t \,.\, t-t_3, \qquad t_2 > t > t_3,$$

(11) $$\frac{a^2-A^2}{MQ^2} = \frac{\sqrt{(\tau-t_2 \,.\, \tau-t_3)}}{\tau-t},$$

(12) $$\frac{a^2-A^2}{MQ^2}\frac{b\,d\theta}{PQ} = \frac{\sqrt{(-U)}}{\tau-t}\cdot\frac{dt}{\sqrt{T}},$$

(13) $$-U = 4 \,.\, t_1-\tau \,.\, \tau-t_2 \,.\, \tau-t_3, \qquad t_1 > \tau > t_2 > t_3,$$

(14) $$\int_0^{2\pi}\frac{a^2-A^2}{MQ^2}\frac{b\,d\theta}{PQ} = \int_{t_3}^{t_2}\frac{2\sqrt{(-U)}}{\tau-t}\cdot\frac{dt}{\sqrt{T}},$$

in a standard Weierstrassian form of the III. E.I.

The expression of this III. E.I. when complete, by means of the E.I., I. and II., complete and incomplete, was given by LEGENDRE, 'Fonctions Elliptiques,' chap. 23, and (14) falls under his class $(m')$.

4. But we shall avoid the Legendrian form, and start by making use of the lemma

(1) $$\sqrt{T}\frac{d}{dt}\frac{\frac{1}{2}\sqrt{T}}{\tau-t} - \sqrt{\phantom{-}}U\frac{d}{d\tau}\frac{\frac{1}{2}\sqrt{-U}}{\tau-t} = \tau-t,$$

proved immediately by effecting the differentiation.

Integrating, in either order, with respect to the differential elements

$$\frac{dt}{\sqrt{T}} \quad\text{and}\quad \frac{d\tau}{\sqrt{-U}}$$

and between the limits $t_2 > t > t_3$, and $t_1$, $\tau$ of $\tau$,

(2) $$\int_\tau^{t_1}\frac{d\tau}{\sqrt{-U}}\int_{t_3}^{t_2}\frac{dt}{\sqrt{T}}\sqrt{T}\frac{d}{dt}\left(\frac{\frac{1}{2}\sqrt{T}}{\tau-t}\right)$$
$$= \int\frac{d\tau}{\sqrt{-U}}\int\frac{d}{dt}\left(\frac{\frac{1}{2}\sqrt{T}}{\tau-t}\right)dt$$
$$= \int\frac{d\tau}{\sqrt{-U}}\left(\frac{\frac{1}{2}\sqrt{T}}{\tau-t}\right)_{t_2}^{t_3} = 0,$$

(3) $$-\int\frac{dt}{\sqrt{T}}\int_\tau^{t_1}\frac{d\tau}{\sqrt{-U}}\sqrt{-U}\frac{d}{d\tau}\left(\frac{\frac{1}{2}\sqrt{-U}}{\tau-t}\right)$$
$$= -\int\frac{dt}{\sqrt{T}}\int\frac{d}{d\tau}\left(\frac{\frac{1}{2}\sqrt{-U}}{\tau-t}\right)d\tau$$
$$= -\int\frac{dt}{\sqrt{T}}\left(\frac{\frac{1}{2}\sqrt{-U}}{\tau-t}\right)_\tau^{t_1}$$
$$= \int_{t_3}^{t_2}\frac{dt}{\sqrt{T}}\frac{\frac{1}{2}\sqrt{-U}}{\tau-t},$$

as in (14), § 3; so calling it 2B, as a standard type of the III. E.I., it is proved by the lemma (1) that

(4)
$$B = \int_{\tau}^{t_1}\int_{t_3}^{t_2} (\tau - t)\frac{dt}{\sqrt{T}}\frac{d\tau}{\sqrt{-U}},$$

in which the variables are separated, $t$ and $\tau$. Then with

(5)
$$G = \int_{t_3}^{t_2}\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}, \qquad G' = \int_{t_2}^{t_1}\frac{\sqrt{(t_1-t_3)}\,d\tau}{\sqrt{-U}},$$

to the modulus $\gamma = \sqrt{\frac{t_2-t_3}{t_1-t_3}}$, and co-modulus $\gamma' = \sqrt{\frac{t_1-t_2}{t_1-t_3}}$, and with $e$ and $f$ to denote fractions, such that

(6)
$$eG = \int_{t_3}^{t}\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}, \qquad 2fG' = \int_{\tau}^{t_1}\frac{\sqrt{(t_1-t_3)}\,d\tau}{\sqrt{-U}},$$

(7)
$$eG = \mathrm{sn}^{-1}\sqrt{\frac{t-t_3}{t_2-t_3}} = \mathrm{cn}^{-1}\sqrt{\frac{t_2-t}{t_2-t_3}} = \mathrm{dn}^{-1}\sqrt{\frac{t_1-t}{t_1-t_3}},$$

(8)
$$2fG' = \mathrm{sn}^{-1}\sqrt{\frac{t_1-\tau}{t_1-t_2}} = \mathrm{cn}^{-1}\sqrt{\frac{\tau-t_2}{t_1-t_2}} = \mathrm{dn}^{-1}\sqrt{\frac{\tau-t_3}{t_1-t_3}},$$

(9)
$$\iint (t-t_3)\frac{dt}{\sqrt{T}}\frac{d\tau}{\sqrt{-U}} = 2fG'\int\frac{t-t_3}{\sqrt{(t_1-t_3)}}\frac{dt}{\sqrt{T}}$$
$$= 2fG'\int_0^1 (1-\mathrm{dn}^2 eG)\,deG = 2fG'(G-H),$$

where H denotes $E(\gamma)$, the complete II. E.I. to modulus $\gamma$;

(10)
$$\iint (\tau-t_3)\frac{d\tau}{\sqrt{-U}}\frac{dt}{\sqrt{T}} = G\int\frac{\tau-t_3}{\sqrt{(t_1-t_3)}}\frac{d\tau}{\sqrt{-U}}$$
$$= G\int_0^{2f}\mathrm{dn}^2 2fG'\,d2fG' = G(2fH'+\mathrm{zn}\,2fG')$$

with $H' = E(\gamma')$; and then

(11)
$$B = G(2fH'+\mathrm{zn}\,2fG')-2fG'(G-H)$$
$$= 2f(GH'+G'H-GG')+G\,\mathrm{zn}\,2fG'$$
$$= \pi f+G\,\mathrm{zn}\,2fG',$$

by LEGENDRE's relation, and this is the equivalent statement of his equation ($m'$), expressed in the notation of JACOBI.

Then L is given by the three E.I.'s in the form

(12)
$$L = \tfrac{1}{2}\pi P\left(a+\frac{A^2}{a}\right)b+\tfrac{1}{2}Mb-2\pi B(a^2-A^2),$$

and so may be said to be expressed in finite terms, that is, by tabulated functions.

5. The various quantities required are shown geometrically on the diagram of fig. 1 and 2. The front aspect is shown in fig. 1 of the circle on AB, and

(1) $$\text{AOQ} = \theta, \qquad \text{ABQ} = \omega, \qquad \frac{\text{EB}}{\text{EA}} = \frac{\text{DB}}{\text{DA}} = \gamma',$$

$$\sin\omega = \sqrt{\frac{t-t_3}{t_2-t_3}} = \text{sn}\, e\text{G}, \qquad \omega = \text{am}\, e\text{G},$$

$$\text{AB}q = \text{AQE} = \text{am}\,(1-e)\,\text{G}, \qquad \text{AQ}q' = \text{am}\,(1+e)\,\text{G},$$

$$\text{EQ} = \text{EA}\,\text{dn}\, e\text{G}, \qquad \text{E}q = \text{EA}\,\text{dn}\,(1-e)\,\text{G}.$$

On fig. 2, where the circle AB is seen edgeways,

(2) $$\frac{\text{PB}}{\text{PA}} = \frac{\text{EB}}{\text{EA}} = \gamma',$$

$$\text{OBP} = \chi, \qquad \sin\chi = \frac{b}{r_2} = \sqrt{\frac{t_1-\tau}{t_1-t_2}} = \text{sn}\, 2f\text{G}', \qquad \chi = \text{am}\, 2f\text{G}',$$

$$\text{OAP} = \chi' = \text{BPF}, \qquad \sin\chi' = \gamma'\sin\chi, \qquad \cos\chi' = \Delta\chi = \text{dn}\, 2f\text{G}',$$

$$\text{P}p = \text{ED}\cos\chi' = \text{ED}\,\text{dn}\, 2f\text{G}'.$$

The circle on ED is orthogonal to the circle on AB when turned round into the same plane as in fig. 1, and in fig. 2 the two circles on AB and ED may be taken to represent the typical electric and magnetic circuit linked together.

6. MAXWELL goes on to show that M is the *stream function* (S.F.) of a (P.F.) *potential function* $\Omega$, such that

(1) $$\frac{d\text{M}}{d\text{A}} = 2\pi\text{A}\frac{d\Omega}{db}, \qquad \frac{d\text{M}}{db} = -2\pi\text{A}\frac{d\Omega}{d\text{A}},$$

(2) $$\frac{d}{d\text{A}}\left(\text{A}\frac{d\Omega}{d\text{A}}\right) + \frac{d}{db}\left(\text{A}\frac{d\Omega}{db}\right) = 0,$$

(3) $$\frac{d}{d\text{A}}\left(\frac{1}{\text{A}}\frac{d\text{M}}{d\text{A}}\right) + \frac{d}{db}\left(\frac{1}{\text{A}}\frac{d\text{M}}{db}\right) = 0,$$

and a line of force along M a constant is at right angles to a surface of constant $\Omega$.

If a return should be made to the usual co-ordinates, it is preferable to employ the ordinary $(x, y)$ of plane geometry, and not the cylindrical or columnar co-ordinates $(z, \varpi)$ or $(z, \rho)$ of some writers, or MAXWELL'S $(b, \text{A})$.

Then these equations (2) and (3) will appear in the familiar form

(4) $$\frac{d}{dx}\left(y\frac{d\Omega}{dx}\right) + \frac{d}{dy}\left(y\frac{d\Omega}{dy}\right) = 0,$$

(5) $$\frac{d}{dx}\left(\frac{1}{y}\frac{d\text{M}}{dx}\right) + \frac{d}{dy}\left(\frac{1}{y}\frac{d\text{M}}{dy}\right) = 0,$$

reducing to ordinary conjugate function relations at a great distance from the axis O$x$, where $y$ is large. And with any conjugate system, $x+iy = f(u+iv)$, $(dx, dy)$ are replaced by $(du, dv)$; thus for polar co-ordinates $x+iy = e^{\log r + i\vartheta}$, $du = dr/r$, $dv = d\vartheta$.

If the motion is not symmetrical about the axis O$x$, and is not uniaxial, the S.F. does not exist; and in equation (4) an additional term is required for the variation with angle $\phi$ of azimuth, so that in this general case

(6) $$\frac{d}{dx}\left(y\frac{d\Omega}{dx}\right)+\frac{d}{dy}\left(y\frac{d\Omega}{dy}\right)+\frac{d^2\Omega}{y\,d\phi^2}=0,$$

expressing the resultant leakage or crowding-convergence of $\Omega$ through an element of volume $dx \,.\, dy/y\, d\phi$, when this is zero.

Thus the result in (7), § 3, that $\frac{\text{M}}{2\pi\text{A}}\cos\phi$ is a P.F., $-\text{Q}\cos\phi$, is true for any S.F. M; for changing A into $y$, and putting $\text{M} = \text{V}y$,

(7) $$\frac{d}{dx}\left(\frac{1}{y}\frac{d\text{M}}{dx}\right)+\frac{d}{dy}\left(\frac{1}{y}\frac{d\text{M}}{dy}\right)$$
$$=\frac{d^2\text{V}}{dx^2}+\frac{d^2\text{V}}{dy^2}+\frac{1}{y}\frac{d\text{V}}{dy}-\frac{\text{V}}{y^2}$$
$$=\frac{1}{y}\left[\frac{d}{dx}\left(y\frac{d\text{V}}{dx}\right)+\frac{d}{dy}\left(y\frac{d\text{V}}{dy}\right)-\frac{\text{V}}{y}\right]=0,$$

so that V $\cos\phi$ satisfies equation (6) for $\Omega$.

7. MAXWELL shows further that $\Omega$ is the magnetic potential of any sheet bounded by the circle AB with uniform normal magnetization, so that, taking the plane circle AB, $\Omega$ is given by the normal component of the surface attraction of the circular disc AB, and so is the solid conical angle subtended at P.

This is true for any boundary AB; for if $d$S denotes any elementary area enclosing a point Q, the element of normal attraction, $\frac{d\text{S}}{\text{PQ}^2}\cos\text{QPM}$, is the element of surface of unit sphere with centre at P, cut out of the cone on the base $d$S.

In MAXWELL'S expression for P, surface potential of a spherical segment on the circular base AB, given in the form of a series of zonal harmonics (E. and M., § 694), he proves that

(1) $$\frac{1}{c}\frac{d}{dr}(\text{P}r)=\Omega,$$

but he does not notice that

(2) $$\text{P}=\Omega c+\Omega'\frac{c^2}{r}=\Omega c+\Omega' r',$$

where $\Omega'$ is the solid angle or apparent area of the circle AB from the inverse point in the sphere.

As the result is independent of the size of the segment, it holds true when the segment is made small, and this can be proved in a couple of lines of elementary geometry, as given in the 'American Journal of Mathematics' (A.J.M.), October, 1917, p. 237. Thence, by summation, the result for P holds in the same form when the spherical segment has any arbitrary boundary not restricted to be circular.

For the analytical expression of $\Omega$ the complete elliptic integral of the third kind (E.I. III.) is required. This is not attempted by MAXWELL, and he leaves $\Omega$ in the form of a series of zonal harmonics obtained and written down from the axial expansion.

But the chief difficulty in the theory of the ampère balance is the reduction and manipulation of $\Omega$, a multiple-valued function with a cyclic constant $4\pi$ for a magnetic circuit through the circle on AB, say round the circle on ED, linked with the electric circuit round AB.

8. The III. E.I. required for $\Omega$ will be of the same nature as B which occurs in L (6), § 2, (14), § 3, and to obtain the relation, take MAXWELL's M and differentiate with respect to A, then

$$(1) \qquad \mathrm{A}\frac{d\Omega}{db} = \frac{1}{2\pi}\frac{d\mathrm{M}}{d\mathrm{A}} = \int \frac{a\cos\theta\, d\theta}{\mathrm{PQ}} - \int \frac{\mathrm{A}a\cos\theta\,(\mathrm{A}+a\cos\theta)\, d\theta}{\mathrm{PQ}^3}.$$

Making use of the lemma,

$$(2) \qquad \int \frac{db}{\mathrm{PQ}^3} = \frac{1}{\mathrm{MQ}^2}\cdot\frac{b}{\mathrm{PQ}},$$

$\Omega$ is obtained by an integration with respect to $b$,

$$(3) \qquad \mathrm{A}\Omega = \int a\cos\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}\,d\theta - \int \frac{\mathrm{A}a\cos\theta\,(\mathrm{A}+a\cos\theta)}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}}.$$

Integrating the first of these integrals by parts,

$$(4) \qquad \mathrm{A}\Omega = \left(a\sin\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}\right)_0^{2\pi} - \int a\sin\theta\,\frac{\mathrm{A}a\sin\theta}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}} - \int \frac{\mathrm{A}a\cos\theta\,(\mathrm{A}+a\cos\theta)}{\mathrm{MQ}^2}\cdot\frac{bd\,\theta}{\mathrm{PQ}}$$

$$= * - \int \frac{\mathrm{A}a\,(\mathrm{A}\cos\theta+a)}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}},$$

the * marking the place of a term which vanishes at both limits,

$$(5) \qquad \Omega = \text{constant} - \int \frac{\mathrm{A}a\cos\theta + a^2}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}}$$

$$= \text{constant} - \Omega\,(\mathrm{MQ}),$$

suppose, where

$$(6) \qquad \Omega\,(\mathrm{MQ}) = \int_0^{2\pi} \frac{\mathrm{A}a\cos\theta + a^2}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}}.$$





PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

In MINCHIN'S dissection of the circle on AB by lines radiating from M, 'Phil. Mag.,' February, 1894, the solid angle cut out by a complete revolution of PQ about PM at a constant angle is $\left(1-\frac{\text{PM}}{\text{PQ}}\right)2\pi$, so that for an elementary angle $d\eta$,

$$(7) \qquad d\Omega = \left(1-\frac{\text{PM}}{\text{PQ}}\right)d\eta, \quad \text{and} \quad \tfrac{1}{2}\text{MQ}^2\,d\eta = \tfrac{1}{2}\text{MY}\,.\,a\,d\theta,$$

if $\text{MY} = \text{A}\cos\theta + a$ is the perpendicular from M on the tangent at Q; so that

$$(8) \qquad d\Omega = d\eta - \frac{\text{A}a\cos\theta + a^2}{\text{MQ}^2}\cdot\frac{b\,d\theta}{\text{PQ}} = d\eta - d\Omega\,(\text{MQ}).$$

In a complete circuit of the circle on AB, $\eta$ grows from 0 to $2\pi$, if M is inside the circle on AB $(a > \text{A}, f < \frac{1}{2})$,

$$(9) \qquad \Omega = 2\pi - \Omega\,(\text{MQ}).$$

Replacing $\text{A}a\cos\theta$ by $\frac{1}{2}(\text{MQ}^2 - a^2 - \text{A}^2)$,

$$(10) \qquad \Omega\,(\text{MQ}) = \tfrac{1}{2}\int_0^{2\pi}\frac{a^2-\text{A}^2}{\text{MQ}^2}\cdot\frac{b\,d\theta}{\text{PQ}} + \tfrac{1}{2}\int\frac{b\,d\theta}{\text{PQ}}$$

$$= 2\text{B} + \frac{2b\text{G}}{r_3}$$

$$= 2\pi f + 2\text{G}\,\text{zn}\,2f\text{G}' + 2\text{G}\gamma'\,\text{sn}\,2f\text{G}',$$

$$(11) \qquad \Omega = 2\pi(1-f) - 2\text{G}\,\text{zn}\,2f\text{G}' - 2\text{G}\gamma'\,\text{sn}\,2f\text{G}'.$$

This agrees in making $\Omega = 2\pi$ when P is at E and AB is viewed close up, and $\Omega = 0$ when $f = 1$ and P is at D, where the circle AB is seen edgeways; and then, with this value of B in (12), § 4,

$$(12) \qquad \text{L} = \pi\text{P}ab + \tfrac{1}{2}\text{M}b - \pi(a^2 - \text{A}^2)(2\pi - \Omega).$$

In making the circuit of the circle EPD, and starting from E, where $f = 0$, $\Omega = 2\pi$, then $f$ grows from 0 to 1, and $\Omega$ diminishes from $2\pi$ to 0 at D. After passing D, $f$ grows from 1 to 2, and $\Omega$ is taken negative for the reverse aspect of the circle AB, and on arrival at E again with $f = 2$, $\Omega = -2\pi$.

Thus $4\pi$ must be added in crossing AB if P circulates counter-clockwise. But with the clock, the other way round, $4\pi$ must be subtracted in crossing AB, just as twelve hours is deducted on the clock in passing through XII o'clock.



PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

9. But proceeding to $\Omega$ through

(1) $$\frac{d\Omega}{dA} = -\frac{1}{2\pi A}\frac{dM}{db} = \int a\cos\theta\,\frac{b\,d\theta}{PQ^3},$$

and utilising the integral

(2) $$\int\frac{dA}{PQ^3} = \frac{A+a\cos\theta}{PR^2}\cdot\frac{1}{PQ}, \qquad PR^2 = a^2\sin^2\theta+b^2,$$

PR the perpendicular from P on QQ′ parallel to AB in fig. 1,

(3) $$\Omega = \Omega\,(PR) = \int\frac{Aa\cos\theta+a^2\cos^2\theta}{PR^2}\cdot\frac{b\,d\theta}{PQ},$$

a new form of the III. E.I., not recognisable in the previous expression in (6), § 8.

We have to make use of the theorems given in the 'Trans. American Math. Society,' 1907 (A.M.S.), connecting the various forms of dissection of $\Omega$ in the III. E.I., and here the relation connecting the incomplete integrals in $\theta$ of $\Omega\,(MQ)$ and $\Omega\,(PR)$ is

(4) $$\Omega\,(MQ)+\Omega\,(PR) = \text{angle between MQP, MQR}$$
$$= \sin^{-1}\frac{QN}{PR}\cdot\frac{PQ}{MQ} = \cos^{-1}\frac{MN}{PR}\cdot\frac{PM}{MQ} = \cot^{-1}\frac{A+a\cos\theta}{a\sin\theta}\cdot\frac{b}{PQ},$$

as is soon verified by the differentiation; and for the complete integrals between 0 and $2\pi$ the sum is $2\pi$.

In $\Omega\,(PR)$ the dissection of the circle AB would be in strips QQ′ parallel to AB.

10. Another form for $\Omega$ is obtained from the theorem that

(1) $$\Phi+\Omega = 2\pi$$

connects $\Omega$, the area of the spherical curve of the cone on the base AB, and $\Phi$, the perimeter of the curve of the reciprocal cone, both on the unit sphere with centre at the vertex P.

The section SS′ of the reciprocal cone made by the plane AB is the polar reciprocal of the circle AB with respect to the pole M; a conic with focus at M, and

(2) $$SM\,.\,MY = ZM\,.\,MQ = b^2.$$

The projection of the elementary sector PSS′ of the reciprocal cone on the plane AB is

(3) $$\tfrac{1}{2}PS^2\,.\,d\Phi\,.\cos PZM = \tfrac{1}{2}MS^2\,d\theta,$$

(4) $$\frac{d\Phi}{d\theta} = \frac{MS^2}{PS^2}\cdot\frac{PQ}{PM} = \frac{MP^2}{PY^2} = \frac{PQ\,.\,b}{PY^2} = \left(1+\frac{QY^2}{PY^2}\right)\frac{b}{PQ}.$$

In the reduction of this form we employ a new substitution, putting

$$(5) \qquad \cos^2 \mathrm{QPY} = \frac{\mathrm{PY}^2}{\mathrm{PQ}^2} = m^2(\sigma - s),$$

where $s$ is the new variable, and $\sigma$ a constant; and take $s = s_3$ at A where QPY = 0, $\cos^2 \mathrm{QPY} = 1 = m^2(\sigma - s_3)$,

$$(6) \qquad \frac{\mathrm{PY}^2}{\mathrm{PQ}^2} = \frac{\sigma - s}{\sigma - s_3}, \qquad \frac{\mathrm{QY}^2}{\mathrm{PQ}^2} = \frac{s - s_3}{\sigma - s_3} = \frac{\mathrm{A}^2 \sin^2\theta}{r^2} = \frac{r_3{}^2 - r^2 \,.\, r^2 - r_2{}^2}{4a^2 r^2},$$

so that $s = \infty$ at $r = 0, \infty$. Then take $s = s_2, s_1$ for $r^2 = \pm r_2 r_3$; this makes

$$(7) \qquad \frac{s_2 - s_3}{\sigma - s_3} = \left(\frac{r_3 - r_2}{2a}\right)^2 = \left(\frac{2\mathrm{A}}{r_3 + r_2}\right)^2, \qquad \frac{s_1 - s_3}{\sigma - s_3} = \left(\frac{r_3 + r_2}{2a}\right)^2 = \frac{2\mathrm{A}}{(r_3 - r_2)^2}.$$

With the variable $s$ we are employing the elliptic function has a new modulus $\kappa$, obtained by a quadric transformation of the former modulus $\gamma$, and associated with the elliptic integral

$$(8) \qquad \mathrm{K} = \int_{s_3}^{s_2} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{\mathrm{S}}}, \qquad \mathrm{K}' = \int_{s_2}^{s_1} \frac{\sqrt{(s_1 - s_3)}\, d\sigma}{\sqrt{-\Sigma}},$$

$$(9) \qquad \mathrm{S} = 4 \,.\, s_1 - s \,.\, s_2 - s \,.\, s - s_3, \qquad -\Sigma = 4 \,.\, s_1 - \sigma \,.\, \sigma - s_2 \,.\, \sigma - s_3,$$

$$(10) \qquad \infty > s_1 > \sigma > s_2 > s > s_3 > -\infty,$$

$$(11) \qquad \kappa^2 = \frac{s_2 - s_3}{s_1 - s_3} = \left(\frac{r_3 - r_2}{r_3 + r_2}\right)^2, \qquad \kappa = \frac{1 - \gamma'}{1 + \gamma'},$$

and with fractions $e$ and $f$, such that

$$(12) \qquad 2e\mathrm{K} = \int_{s_3}^{s} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{\mathrm{S}}}, \qquad f\mathrm{K}' = \int_{\sigma}^{s_1} \frac{\sqrt{(s_1 - s_3)}\, d\sigma}{\sqrt{-\Sigma}},$$

$$(13) \qquad 2e\mathrm{K} = \mathrm{sn}^{-1}\sqrt{\frac{s - s_3}{s_2 - s_3}} = \mathrm{cn}^{-1}\sqrt{\frac{s_2 - s}{s_2 - s_3}} = \mathrm{dn}^{-1}\sqrt{\frac{s_1 - s}{s_1 - s_3}},$$

$$(14) \qquad f\mathrm{K}' = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - \sigma}{s_1 - s_2}} = \mathrm{cn}^{-1}\sqrt{\frac{\sigma - s_2}{s_1 - s_2}} = \mathrm{dn}^{-1}\sqrt{\frac{\sigma - s_3}{s_1 - s_3}}.$$

Produce PE on fig. 2 to meet $Ox$ in G, and describe the sphere, centre G, passing through the circle on AB. Then P, E are inverse points in this sphere, and

(15) $$\frac{PQ}{EQ} = \frac{PA}{EA} = \frac{PB}{EB} = \frac{PA+PB}{AB} = \frac{r_3+r_2}{2a},$$

(16) $$\text{sn}^2\, 2eK = \frac{s-s_3}{s_2-s_3} = \frac{s-s_3}{\sigma-s_3}\cdot\frac{\sigma-s_3}{s_2-s_3} = \frac{A^2\sin^2\theta}{r^2}\left(\frac{r_3+r_2}{2A}\right)^2,$$

(17) $$\text{sn}\, 2eK = \frac{r_3+r_2}{2PQ}\sin\theta = \frac{a\sin\theta}{EQ} = \sin\phi, \qquad \phi = \text{am}\, 2eK,$$

where $\phi$ = AEQ on fig. 1, and

(18) $$\kappa = \frac{OE}{OB} = \frac{OB}{OD} = \frac{BE}{BD}.$$

And with ODQ = $\phi'$ = OQE on fig. 1,

(19) $$\sin\phi' = \kappa\sin\phi, \qquad \cos\phi' = \Delta\phi = \text{dn}\, 2eK,$$

so that $Qq = AB\cos\phi' = AB\,\text{dn}\, 2eK$.

On fig. 2, and from (7), (14),

(20) $$\text{dn} f K' = \sqrt{\frac{\sigma-s_3}{s_1-s_3}} = \frac{2a}{r_3+r_2} = \frac{EB}{PB} = \frac{Bp}{BD},$$

and with DE$p$ = DPB = $\psi$,

(21) $$Bp^2 = BD^2\cos^2\psi + BE^2\sin^2\psi = BD^2\Delta^2(\psi, \kappa'),$$

(22) $$\Delta(\psi, \kappa') = \text{dn} f K', \qquad \psi = \text{am} f K',$$

and

$$DEP = DpB = \text{am}\,(1-f)\,K'.$$

Thence any formula of the Landen quadric transformation, first and second, can be interpreted geometrically on fig. 1 and 2, and we reconcile the baffling and conflicting notation of previous writers on the subject.

Interpreted dynamically, with $e$ proportional to the time, $t = \frac{1}{2}e\mathrm{T}$ for period T, Q circulates round the circle on AB in fig. 1 with velocity due to the level of F, or proportional to EQ or DQ, and beats the elliptic function to modulus $\gamma$, while T circulates round the circle on OE with velocity due to the level of D, or S at the same level will oscillate, beating elliptic functions to modulus $\kappa$.

So also for the motion of P round the circle on ED, with $f$ proportional to the time $t$, and velocity due to the level of O, or proportional to BP or AP, gravity being reversed.

11. Combined into one quadric transformation, the first and second of Landen, from modulus $\gamma$ to $\kappa$, and then $\kappa$ to $\gamma$,

$$(1)\qquad \sqrt{\kappa}\,\mathrm{sn}\,(2e\mathrm{K}+f\mathrm{K}'i) = \frac{\gamma\,\mathrm{sn}\,(e\mathrm{G}+f\mathrm{G}'i)\,\mathrm{cn}\,(e\mathrm{G}+f\mathrm{G}'i)}{\mathrm{dn}\,(e\mathrm{G}+f\mathrm{G}'i)}$$

$$= \gamma\,\mathrm{sn}\,(e\mathrm{G}+f\mathrm{G}'i)\,\mathrm{sn}\,(\mathrm{G}-e\mathrm{G}-f\mathrm{G}'i),$$

$$(2)\qquad \mathrm{dn}\,(2e\mathrm{G}+2f\mathrm{G}'i) = \frac{1-\kappa\,\mathrm{sn}^2\,(2e\mathrm{K}+f\mathrm{K}'i)}{1+\kappa\,\mathrm{sn}^2\,(2e\mathrm{K}+f\mathrm{K}'i)},$$

and then $f$ or $e$ is made zero. And

$$(3)\qquad \mathrm{cn}\,(2e\mathrm{K}+f\mathrm{K}'i) = \frac{\mathrm{dn}^2\,(e\mathrm{G}+f\mathrm{G}'i)-\gamma'}{(1-\gamma')\,\mathrm{dn}\,(e\mathrm{G}+f\mathrm{G}'i)}$$

$$= \frac{\mathrm{dn}\,(e\mathrm{G}+f\mathrm{G}'i)-\mathrm{dn}\,(\mathrm{G}-e\mathrm{G}-f\mathrm{G}'i)}{1-\gamma'},$$

$$(4)\qquad \mathrm{dn}\,(2e\mathrm{K}+f\mathrm{K}'i) = \frac{\mathrm{dn}^2\,(e\mathrm{G}+f\mathrm{G}'i)+\gamma'}{(1+\gamma')\,\mathrm{dn}\,(e\mathrm{G}+f\mathrm{G}'i)}$$

$$= \frac{\mathrm{dn}\,(e\mathrm{G}+f\mathrm{G}'i)+\mathrm{dn}\,(\mathrm{G}-e\mathrm{G}-f\mathrm{G}'i)}{1+\gamma'},$$

$$(5)\qquad \frac{\kappa}{\kappa'}\,\mathrm{cn}\,(2e\mathrm{K}+f\mathrm{K}'i) = \tfrac{1}{2}\,\frac{\mathrm{dn}\,(e\mathrm{G}+f\mathrm{G}'i)}{\sqrt{\gamma'}} - \tfrac{1}{2}\,\frac{\mathrm{dn}\,(\mathrm{G}-e\mathrm{G}-f\mathrm{G}'i)}{\sqrt{\gamma'}},$$

$$(6)\qquad \frac{\mathrm{dn}\,(2e\mathrm{K}+f\mathrm{K}'i)}{\kappa'} = \tfrac{1}{2}\,\frac{\mathrm{dn}\,(e\mathrm{G}+f\mathrm{G}'i)}{\sqrt{\gamma'}} + \tfrac{1}{2}\,\frac{\mathrm{dn}\,(\mathrm{G}-e\mathrm{G}-f\mathrm{G}'i)}{\sqrt{\gamma'}}.$$

12. By logarithmic differentiation of (6), § 10,

(1) $$\frac{ds}{s-s_3} = \frac{-r^4+r_2^{\,2}r_3^{\,2}}{r_3-r^2 \,.\, r^2-r_2^{\,2}} \frac{dr^2}{r^2},$$

and with $s = s_2, s_1$, for $r^2 = \pm\, r_2 r_3$,

(2) $$\frac{s_1-s}{\sigma-s_3} = \left(\frac{r^2+r_2r_3}{2ar}\right)^2, \qquad \frac{s_2-s}{\sigma-s_3} = \left(\frac{r^2-r_2r_3}{2ar}\right)^2,$$

(3) $$\frac{\sqrt{(s_1-s \,.\, s_2-s)}}{\sigma-s_3} = \frac{r^4-r_2^{\,2}r_3^{\,2}}{4a^2r^2},$$

(4) $$\sqrt{\frac{s-s_3}{\sigma-s_3}} = \frac{\sqrt{(r_3^{\,2}-r^2 \,.\, r^2-r_2^{\,2})}}{2ar},$$

(5) $$\frac{\sqrt{\mathrm{S}}}{(\sigma-s_3)^3} = \frac{(r^4-r_2^{\,2}r_3^{\,2})\sqrt{(r_3^{\,2}-r^2 \,.\, r^2-r_2^{\,2})}}{4a^3r^3},$$

(6) $$\frac{ds}{\sigma-s_3} = (r^4-r_2^{\,2}r_3^{\,2})\frac{-2\,dr}{4a^2r^3},$$

(7) $$\frac{\sqrt{(\sigma-s_3)}\,ds}{\sqrt{\mathrm{S}}} = \frac{-2a\,dr}{\sqrt{(r_3^{\,2}-r^2 \,.\, r^2-r_2^{\,2})}}.$$

Then with

(8) $$\theta = 2\sin^{-1}\sqrt{\frac{r_3^{\,2}-r^2}{r_3^{\,2}-r_2^{\,2}}} = 2\cos^{-1}\sqrt{\frac{r^2-r_2^{\,2}}{r_3^{\,2}-r_2^{\,2}}}, \qquad d\theta = \frac{-2r\,dr}{\sqrt{(r_3^{\,2}-r^2 \,.\, r^2-r_2^{\,2})}},$$

(9) $$d\mathrm{P} = \frac{a\,d\theta}{\mathrm{PQ}} = \frac{-2a\,dr}{\sqrt{(r_3^{\,2}-r^2 \,.\, r^2-r_2^{\,2})}} = \frac{\sqrt{(\sigma-s_3)}\,ds}{\sqrt{\mathrm{S}}}$$
$$= \sqrt{\frac{\sigma-s_3}{s_1-s_3}}\,d2e\mathrm{K} = \mathrm{dn} f\,\mathrm{K}'\,d2e\mathrm{K} = \frac{\mathrm{EB}}{\mathrm{PB}}\,d2e\mathrm{K}.$$

Comparing this with the previous form in (4), § 3,

(10) $$d\mathrm{P} = \frac{2a}{r_3}\frac{d\omega}{\Delta\omega} = \frac{\mathrm{AB}}{\mathrm{AP}}\,de\mathrm{G},$$

(11) $$\frac{\mathrm{EB}}{\mathrm{PB}}\,2\mathrm{K} = \frac{\mathrm{AB}}{\mathrm{AP}}\,\mathrm{G}, \qquad \frac{2\mathrm{K}}{\mathrm{G}} = \frac{\mathrm{AB}}{\mathrm{EB}} \cdot \frac{\mathrm{PB}}{\mathrm{AP}} = \frac{2\gamma'}{1-\kappa}.$$

(12) $$(1-\kappa)\,\mathrm{K} = \mathrm{G}\gamma', \qquad \mathrm{K}\kappa' = \mathrm{G}\sqrt{\gamma'}.$$

And similarly

(13) $$(1-\kappa)\,K' = 2G'\gamma', \qquad \frac{K'}{K} = 2\,\frac{G'}{G},$$

as in the quadric transformation. Thus

(14) $$P = 4K\,\mathrm{dn}f K' = \frac{8aK}{r_2+r_3} = \frac{4aK\kappa'}{\sqrt{(r_2r_3)}}, \quad \text{or} \quad \frac{4aG}{r_3} = \frac{4aG\sqrt{\gamma'}}{\sqrt{(r_2r_3)}}.$$

On fig. 1, $\theta = \phi+\phi'$,

(15) $$\cos\theta = \cos\phi\cos\phi' - \sin\phi\sin\phi'$$
$$= \mathrm{cn}\,2eK\,\mathrm{dn}\,2eK - \kappa\,\mathrm{sn}^2\,2eK,$$

(16) $$dQ = \frac{-a\cos\theta\,d\theta}{PQ} = \mathrm{dn}f K'(\kappa\,\mathrm{sn}^2\,2eK - \mathrm{cn}\,2eK\,\mathrm{dn}\,2eK)\,d2eK,$$

and integrating round AB from $0 < e < 2$, the second term in $dQ$ vanishes by inspection, and

(17) $$Q = 2\,\frac{\mathrm{dn}\,fK'}{\kappa}\int_0^2 (1-\mathrm{dn}^2\,2eK)\,d2eK = 4\,\frac{K-E}{\mathrm{dn}\,(1-f)\,K'}$$
$$= \frac{r_3+r_2}{2A}\,4\,(K-E) = \frac{4a}{\sqrt{(r_2r_3)}}\,(K-E)\,\frac{\kappa'}{\kappa},$$

(18) $$M = -2\pi QA = -4\pi\,(r_2+r_3)\,(K-E) = -2\pi\sqrt{(r_2r_3)}\,\frac{K-E}{\kappa'}.$$

Thus in the construction of the curves of constant M on the Weir chart, a table was first drawn up from LEGENDRE of E and K for every degree of the modular angle $\theta$, and then of K$-$E and $\dfrac{K-E}{\sin\theta}$; and with the hour angle $\alpha = \frac{1}{2}\pi-\xi$, and $\lambda$ the latitude, such that $\mathrm{ch}\,\eta = \sec\lambda$, and $r_3$, $r_2 = c\,(\mathrm{ch}\,\eta \pm \cos\xi)$,

(19) $$N = -\frac{M}{8\pi c}\,\mathrm{ch}\,\eta\,(K-E), \qquad \cos\lambda = \frac{K-E}{N},$$

(20) $$\sin\theta = \frac{r_3-r_2}{r_3+r_2} = \frac{\cos\xi}{\mathrm{ch}\,\eta} = \sin\alpha\cos\lambda, \qquad \sin\alpha = \frac{\sin\theta}{\cos\lambda} = \frac{N}{\dfrac{K-E}{\sin\theta}},$$

whence $\lambda$, $\alpha$ were calculated for given N, starting from $\lambda = 0$, when $N = K-E$, $\alpha = \theta$

Another method is given by MAXWELL in 'E. and M.,' § 702.

13. Next, for $\Phi$ and $\Omega$,

(1) $$d\Phi = \frac{PQ^2}{PY^2} \cdot \frac{b\,d\theta}{PQ} = \frac{\sigma - s_3}{\sigma - s} \cdot \frac{b}{a} \cdot \frac{\sqrt{(\sigma - s_3)}\,ds}{\sqrt{S}},$$

(2) $$\frac{s_1 - \sigma}{\sigma - s_3} = \left(\frac{r_3 + r_2}{2a}\right)^2 - 1, \qquad \frac{\sigma - s_2}{\sigma - s_3} = 1 - \left(\frac{r_3 - r_2}{2a}\right)^2, \qquad \frac{\sqrt{(s_1 - \sigma \,.\, \sigma - s_2)}}{\sigma - s_3} = \frac{b}{a},$$

(3) $$d\Phi = \frac{\frac{1}{2}\sqrt{-\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}},$$

(4) $$\Phi = 4\int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{-\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}} = 2\pi f + 4K\,\mathrm{zn}\,fK',$$

in accordance with the previous expression for B, or $B(2fG')$, and

(5) $$\Omega = 2\pi - \Phi = 2\pi(1-f) - 4K\,\mathrm{zn}\,fK'.$$

Comparing this with the previous expression for $\Omega$ in (11), § 8, we have the theorem of the quadric transformation of the zeta function

(6) $$2K\,\mathrm{zn}\,fK' = G\,\mathrm{zn}\,2fG' + G\gamma'\,\mathrm{sn}\,2fG'.$$

This is obtained by taking the quadric transformation formula, obtained from the geometry of fig. 2,

(7) $$\mathrm{dn}\,fK' = \frac{\mathrm{dn}\,2fG' + \gamma'\,\mathrm{cn}\,2fG'}{1+\gamma'}, \quad \text{or} \quad 2K\,\mathrm{dn}\,fK' = G\,\mathrm{dn}\,2fG' + G\gamma'\,\mathrm{cn}\,2fG',$$

squaring it, and integrating both sides with respect to $f$.

According as the modulus $\gamma$ or $\kappa$ is employed, connected by the quadric transformation, as in MAXWELL's 'E. and M.,' § 702, we take, to the modulus $\gamma$,

(8) $$P = \frac{4aG}{r_3} = \frac{4aG\sqrt{\gamma'}}{\sqrt{(r_2 r_3)}}, \qquad P\frac{b}{a} = 4G\gamma'\,\mathrm{sn}\,2fG',$$

$$M = -2\pi r_3[(2-\gamma^2)G - 2H],$$

$$\Omega(f) = \Omega = 2\pi(1-f) - 2G\,\mathrm{zn}\,2fG' - 2G\gamma'\,\mathrm{sn}\,2fG',$$

$$\Omega(f) + \Omega(1-f) = 2\pi - P\frac{b}{a},$$

$$\mathrm{sn}\,2fG' = \frac{b}{r_2} = \frac{MP}{PB}, \qquad \mathrm{cn}\,2fG' = \frac{a - A}{r_2} = \frac{MB}{PB},$$

$$\mathrm{dn}\,2fG' = \frac{a + A}{r_3} = \frac{Pp}{ED} = \cos BAP = \cos FPp', \qquad \gamma' = \frac{PB}{PA},$$

or, to the modulus $\kappa$,

$$(9)\qquad \mathrm{P} = \frac{8a\mathrm{K}}{r_2+r_3} = \frac{4a\mathrm{K}\kappa'}{\sqrt{(r_2r_3)}}, \qquad \mathrm{P}\frac{b}{a} = 4\mathrm{K}\kappa'^2 \,\mathrm{sn}\, f\mathrm{K}' \,\mathrm{sn}\,(1-f)\,\mathrm{K}',$$

$$\mathrm{M} = -2\pi(r_3-r_2)(\mathrm{K}-\mathrm{E}) = -4\pi\sqrt{(r_2r_3)}(\mathrm{K}-\mathrm{E})\frac{\kappa}{\kappa'},$$

$$\Omega = 2\pi(1-f) - 4\mathrm{K}\,\mathrm{zn}\, f\mathrm{K}'.$$

$$\kappa = \frac{r_3-r_2}{r_3+r_2} = \frac{\mathrm{OE}}{\mathrm{OB}} = \frac{\mathrm{OB}}{\mathrm{OD}} \doteq \frac{\mathrm{BE}}{\mathrm{BD}},$$

$$\mathrm{DPB} = \mathrm{am}\, f\mathrm{K}', \qquad \mathrm{DEP} = \mathrm{am}\,(1-f)\,\mathrm{K}',$$

$$\mathrm{dn}\, f\mathrm{K}' = \frac{\mathrm{B}p}{\mathrm{BD}}.$$

The article in the 'Trans. American Math. Society,' October, 1907 (A.M.S.), may be consulted for an elaborate and detailed discussion of the elliptic function analysis and procedure of former writers, and a numerical calculation is given there for the helix employed originally by VIRIAMU JONES. Measurement of fig. 2 gives $\kappa$, $\psi$, and thence, from LEGENDRE'S tables, K, E, $\mathrm{F}\psi$, and $f = \mathrm{F}\psi/\mathrm{K}'$.

Another numerical application of these formulas can be chosen from the dimensions of the current weigher at the N.P.L., Teddington, described in 'Phil. Trans.,' 1907.

14. Integrate $\Omega$ with respect to $b$ to obtain the magnetic potential of the solenoidal current sheet, or of the equivalent close helical winding in the ampère balance.

In these integrations with respect to $b$ the form $\Omega$ (MQ) of the III. E.I. comes in most appropriate as not involving $b$ in MQ, and then

$$(1)\qquad \int \Omega\, db = 2\pi b - \iint \frac{\mathrm{A}a\cos\theta + a^2}{\mathrm{MQ}^2}\frac{b\, d\theta\, db}{\mathrm{PQ}}$$

$$= 2\pi b - \int \frac{\mathrm{A}a\cos\theta + a^2}{\mathrm{MQ}^2}\,\mathrm{PQ}\, d\theta$$

$$= 2\pi b - \int (\mathrm{A}a\cos\theta + a^2)\left(1 + \frac{b^2}{\mathrm{MQ}^2}\right)\frac{d\theta}{\mathrm{PQ}}$$

$$= 2\pi b - \frac{\mathrm{M}}{2\pi} - \mathrm{P}a - b\Omega\,(\mathrm{MQ})$$

$$= -\mathrm{P}a + \mathrm{QA} + \Omega b.$$

This solenoidal magnetic potential is the same as that of the cylinder on which the helix is wound, and so is the equivalent of the axial component of the gravitation attraction of the solid cylinder, and this is the difference of the potentials of the end circular plates. In this way we have arrived at the expression for W, the surface potential of the circle on AB, in the form

(2) $$W = Pa - QA - \Omega b,$$

(3) $$\left(\frac{dW}{da}, \frac{dW}{dA}, \frac{dW}{db}\right) = P, \ -Q, \ -\Omega,$$

in accordance with EULER's theorem for a homogeneous function, in this case W is a homogeneous function of the first degree in the three variables $a$, A, $b$.

At a meeting of the London Mathematical Society, November 11, 1869 ('Proc. L.M.S.,' vol. III, p. 8), Prof. CAYLEY presiding, the Secretary, Mr. JENKINS, read a letter from Mr. CLERK MAXWELL asking the following question: "Can the potential of a uniform circular disc at any point be expressed by means of elliptic integrals?—I am writing out the theory of circular currents in which such quantities occur."

But the result is obvious from the theorem above of a homogeneous function, so that

(4) $$W = \frac{dW}{da}a + \frac{dW}{dA}A + \frac{dW}{db}b,$$

in which

(5) $$\frac{dW}{da} = P = \int \frac{a\,d\theta}{PQ}, \quad \frac{dW}{dA} = \int \frac{+a\cos\theta\,d\theta}{PQ} = -Q, \quad \text{and} \quad \frac{dW}{db} = -\Omega,$$

for any shape of the disc.

Prof. CAYLEY's attention was thereby directed to the subject, and he extended the investigation to the elliptic disc ('L.M.S.,' vol. VI).

15. Integrate P with respect to $b$ to obtain the skin P.F. of the curved surface of the cylinder, drawing out the circle on AB like a concertina,

(1) $$\int P\,db = \iint \frac{a\,d\theta\,db}{PQ} = \int \text{th}^{-1}\frac{b}{PQ}\,a\,d\theta = I,$$

suppose, an intractable integral, $\text{th}^{-1}(b/PQ)$ being the potential of the generating line element of length $b$.

But $\int \cos\theta\,\text{th}^{-1}\frac{b}{PQ}\,d\theta$, as in the expression for L in § 2 (2), is tractable and given in finite terms, while $\int \sin\theta\,\text{th}^{-1}\frac{b}{PQ}\,d\theta$ is non-elliptic, expressed in the variable $\cos\theta$.

The integral I cannot be made to depend on a finite number of elliptic integrals, but requires to be expanded in an infinite series, and so we say it cannot be expressed in finite terms.

Expanded in a series

(2) $$\text{th}^{-1}\frac{b}{PQ} = \Sigma\,\frac{1}{2n+1}\left(\frac{b}{PQ}\right)^{2n+1} = \Sigma\,\frac{1}{2n+1}\left(\frac{b}{r_3}\right)^{2n+1}\frac{1}{(\Delta\omega)^{2n+1}},$$

where, as before, in (3), § 3,

(3) $$\theta = 2\omega, \qquad r^2 = r_3^{\,2}\cos^2\omega + r_2^{\,2}\sin^2\omega = r_3^{\,2}\,\Delta^2(\omega,\gamma), \qquad \gamma' = \frac{r_2}{r_3},$$

(4) $$\int_0^{2\pi}\text{th}^{-1}\frac{b}{PQ}\,.\,a\,d\theta = \Sigma\,\frac{1}{2n+1}\left(\frac{b}{r_3}\right)^{2n+1}\int_0^{\pi}\frac{2a\,d\omega}{(\Delta\omega)^{2n+1}} = 4a\Sigma\,\frac{1}{2n+1}\left(\frac{b^2}{r_2r_3}\right)^{n+\frac{1}{2}}P_n(u),$$

where $P_n(u)$ is the toroidal function, introduced by W. M. Hicks, 'Phil. Trans.,' 1881–4, defined by

(5) $$P_n(u) = \int_0^{\pi}\frac{d\theta}{(\text{ch}\,u+\text{sh}\,u\cos\theta)^{n+\frac{1}{2}}} = \int_0^{\pi}\left(\frac{EA\,.\,EB}{EQ^2}\right)^{n+\frac{1}{2}}d\vartheta$$

$$= 2\sqrt{\gamma'}\int_0^{G}\left(\frac{\text{dn}^2 v}{\gamma'}\right)^n dv, \qquad \gamma' = e^{-u},$$

given by the substitution

(6) $$\text{ch}\,u+\text{sh}\,u\cos\theta = \frac{\text{dn}^2 v}{\gamma'} = \frac{EQ^2}{EA\,.\,EB} = \frac{EA\,.\,EB}{Eq^2}, \qquad \tfrac{1}{2}\theta = \text{am}\,(v,\gamma);$$

and $P_n$ satisfies the differential equation

(7) $$\frac{d^2P}{du^2} + \coth u\,\frac{dP}{du} = \frac{d}{dC}(C^2-1)\frac{dP}{dC} = (n^2-\tfrac{1}{4})\,P,$$

with $C = \text{ch}\,u$, and the sequence difference equation

(8) $$(2n+1)\,P_{n+1} - 4nCP_n + (2n-1)\,P_{n-1} = 0.$$

Expressed otherwise, with $u = eG$, $v = G+2fG'i$,

(9) $$\text{sn}\,v = \frac{r_3}{a+A}, \qquad \text{cn}\,v = \frac{ib}{a+A}, \qquad \text{dn}\,v = \frac{a-A}{a+A}, \qquad \theta = 2\,\text{am}\,u,$$

(10) $$\text{th}^{-1}\frac{b}{PQ} = \text{th}^{-1}\frac{i\,\text{cn}\,v}{\text{sn}\,v\,\text{dn}\,u} = i\tan^{-1}\frac{\text{cn}\,v}{\text{sn}\,v\,\text{dn}\,u} = \tfrac{1}{2}i\,[\text{am}\,(u-v) - \text{am}\,(u+v)],$$

(11) $$I = 2ai\int_0^{G}[\text{am}\,(u-v) - \text{am}\,(u+v)]\,d\,\text{am}\,u.$$

16. The S.F. of the P.F. W is L, so that the S.F. of P is $\frac{dL}{da}$; and in (2), § 2, with

(1) $$\frac{d}{da}\,\mathrm{th}^{-1}\frac{b}{PQ} = -\frac{A\cos\theta + a}{MQ^2}\cdot\frac{b}{PQ},$$

(2) $$\frac{dL}{da} = \int 2\pi A\cos\theta\,\mathrm{th}^{-1}\frac{b}{PQ}\,d\theta - \int 2\pi Aa\cos\theta\,\frac{A\cos\theta+a}{MQ^2}\cdot\frac{b\,d\theta}{PQ},$$

(3) $$a\frac{dL}{da} = L - \int 2\pi\,\frac{A^2a^2\cos^2\theta + Aa^2\cos\theta}{MQ^2}\cdot\frac{b\,d\theta}{PQ}$$

$$= -2\pi\int\frac{A^2a^2 + Aa^2\cos\theta}{MQ^2}\,\frac{b\,d\theta}{PQ}$$

$$= -2\pi a^2\int\frac{MQ^2 - Aa\cos\theta - a^2}{MQ^2}\,\frac{b\,d\theta}{PQ}$$

$$= -2\pi Pab + 2\pi a^2\Omega\,(MQ),$$

(4) $$\frac{dL}{da} = 2\pi a\,(2\pi - \Omega) - 2\pi Pb = 2\pi a\Omega\,(1-f),$$

is the S.F. of P, the rim potential of the circle AB, and L in (12), § 8, is the S.F. of the circular disc on AB. But then, from (8), § 13, $\Omega\,(1-f)$ is the solid angle of the circle on the radius NP seen from Q on the circle on AB.

The S.F. of I, P.F. of the cylindrical skin, is then given by

(5) $$J = \int\frac{dL}{da}\,db = 2\pi a\int\frac{-MQ^2 + Aa\cos\theta + a^2}{MQ^2}\,PQ\,d\theta$$

$$= \pi a\int(-MQ^2 + a^2 - A^2)\left(1 + \frac{b^2}{MQ^2}\right)\frac{d\theta}{PQ}$$

$$= \pi a\int(-MQ^2 + a^2 - A^2 - b^2)\,\frac{d\theta}{PQ} + \pi ab\int\frac{a^2 - A^2}{MQ_2}\cdot\frac{b\,d\theta}{PQ}$$

$$= \pi a\int(-2A^2 - b^2 - 2Aa\cos\theta)\,\frac{d\theta}{PQ} + 4\pi abB$$

$$= -\pi P\,(2A^2 + b^2) - Ma + 2\pi ab\Omega\,(MQ) - \pi Pb^2$$

$$= -2\pi P\,(A^2 + b^2) + 2\pi QAa + 2\pi ab\,(2\pi - \Omega),$$

(6) $$\frac{J}{2\pi} - Wa = -P\,(a^2 + A^2 + b^2) + 2QAa + 2\pi ab,$$

so that J is given in finite terms, while I is intractable, and requires to be given in a series. And generally in these investigations we find the S.F. has the superiority over the P.F. in simplicity of analytical structure. Thus the S.F. at P of the rod AB is PA$-$PB, and of the electrified disc AB is $\sqrt{[AB^2 - (PA - PB)^2]}$.

17. The P.F. of the solid cylinder is given by

(1) $$V = \int W\,db,$$

in which

(2) $$\int Pa\,db = \iint \frac{a^2\,d\theta\,db}{PQ} = \int \text{th}^{-1}\frac{b}{PQ}\,a^2\,d\theta = aI,$$

(3) $$\int -QA\,db = \iint Aa\cos\theta\,\frac{d\theta\,db}{PQ} = -\frac{L}{2\pi},$$

(4) $$\int -\Omega b\,db = \int [\Omega\,(MQ)-2\pi]\,b\,db$$
$$= \iint \frac{Aa\cos\theta + a^2}{MQ^2}\,\frac{b^2\,d\theta\,db}{PQ} - \pi b^2,$$

bringing in again the same intractable integral I.

We obtain V otherwise from the integration of

(5) $$\frac{dV}{da} = I.$$

As a verification we have to prove by differentiation that

(6) $$\frac{1}{2\pi A}\frac{dJ}{dA} = \frac{dI}{db} = P,$$

implied in the integration of (1), § 15, and

(7) $$\frac{dJ}{db} = -2\pi A\,\frac{dI}{dA} = \frac{dL}{da} = -2\pi Pb + 2\pi a\,(2\pi-\Omega) = 2\pi a\Omega\,(1-f),$$

implied in (4), § 16, the expression of the rim S.F. of the circle AB.

And for the P.F. W and its S.F. $\frac{dL}{da}$

(8) $$2\pi A\,\frac{dW}{dA} = -2\pi QA\,\frac{d^2L}{da\,db},$$

(9) $$-2\pi A\,\frac{dW}{db} = 2\pi\Omega A = \frac{d^2L}{da\,dA}.$$

18. An integration of L in (6), § 2, with respect to $b$ will give the S.F. of the solid cylinder

(1) $$N = \int L\,db = \tfrac{1}{2}\pi \int \left[a^2 + A^2 - 2Aa\cos\theta - \frac{(a^2-A^2)^2}{MQ^2}\right] PQ\,d\theta,$$

and then $\frac{dN}{da}$ should be J, the S.F. of the cylindrical skin, as a verification. Here

(2) $$\int \mathrm{PQ}\, d\theta = \mathrm{P}\frac{\mathrm{A}^2+a^2+b^2}{a} - 2\mathrm{QA},$$

(3) $$\int \mathrm{PQ}\cos\theta\, d\theta = \tfrac{2}{3}\mathrm{PA} - \tfrac{1}{3}\mathrm{Q}\frac{\mathrm{A}^2+a^2+b^2}{a},$$

(4) $$\int \frac{(a^2-\mathrm{A}\,)^2}{\mathrm{MQ}^2}\mathrm{PQ}\, d\theta = \int (a^2-\mathrm{A}^2)^2\left(1+\frac{b^2}{\mathrm{MQ}^2}\right)\frac{d\theta}{\mathrm{PQ}}$$
$$= (a^2-\mathrm{A}^2)\left(\mathrm{P}\frac{\mathrm{A}^2-a^2}{a}+4\mathrm{B}b\right),$$

(5) $$\mathrm{N} = \tfrac{4}{3}\pi\mathrm{P}a\mathrm{A}^2 + \tfrac{1}{2}\pi\mathrm{P}\frac{b^2}{a}(a^2+\mathrm{A}^2) - \tfrac{1}{3}\pi\mathrm{QA}(2a^2+2\mathrm{A}^2-b^2) - 2\pi\mathrm{B}b(a^2-\mathrm{A}^2)$$
$$= \pi\mathrm{P}a(\tfrac{4}{3}\mathrm{A}^2+b^2) - \tfrac{1}{3}\pi\mathrm{QA}(2a^2+2\mathrm{A}^2-b^2) - \pi(a^2-\mathrm{A}^2)\,b\Omega\,(\mathrm{MQ}).$$

In the interior of the solid cylinder of unit density, LAPLACE's equation (2), § 6, changes to

(6) $$\frac{d}{d\mathrm{A}}\left(\mathrm{A}\frac{d\mathrm{V}}{d\mathrm{A}}\right) + \frac{d}{db}\left(\mathrm{A}\frac{d\mathrm{V}}{db}\right) + 4\pi\mathrm{A} = 0,$$

or with $\mathrm{V}' = \mathrm{V}+\pi\mathrm{A}^2$,

(7) $$\frac{d}{d\mathrm{A}}\left(\mathrm{A}\frac{d\mathrm{V}'}{d\mathrm{A}}\right) + \frac{d}{db}\left(\mathrm{A}\frac{d\mathrm{V}'}{db}\right) = 0,$$

so that the S.F. N′ is given by

(8) $$\frac{d\mathrm{N}'}{d\mathrm{A}} = 2\pi\mathrm{A}\frac{d\mathrm{V}'}{db} = 2\pi\mathrm{A}\frac{d\mathrm{V}}{db},$$
$$\frac{d\mathrm{N}'}{db} = -2\pi\mathrm{A}\frac{d\mathrm{V}'}{d\mathrm{A}} = -2\pi\mathrm{A}\left(\frac{d\mathrm{V}}{d\mathrm{A}}+2\pi\mathrm{A}\right)$$
$$= -2\pi\mathrm{A}\frac{d\mathrm{V}}{d\mathrm{A}} - 4\pi^2\mathrm{A}^2,$$

requiring the subtraction of $4\pi^2\mathrm{A}^2$ in the interior volume.

19. With these values of a P.F. and its S.F. the relations must verify in § 6 (1, 2, 3).

Thus for the P.F. V and S.F. N of the solid cylinder,

(1) $$\frac{1}{2\pi\mathrm{A}}\frac{d\mathrm{N}}{d\mathrm{A}} = \frac{d\mathrm{V}}{db} = \mathrm{W},$$

implied in the integration in (1), § 18,

(2) $$\frac{d\mathrm{N}}{db} = -2\pi\mathrm{A}\frac{d\mathrm{V}}{d\mathrm{A}} = \mathrm{L}.$$

In making these verifications use must be made of the differentiation formulas given in the 'American Journal of Mathematics' ('A.J.M.'), 1910, p. 392, where D denoting $r_2^2 r_3^2 = (A^2+a^2+b^2)^2-4A^2a^2$,

(3) $$\frac{dP}{dA} = -PA\frac{A^2-a^2+b^2}{D}+Qa\frac{A^2-a^2-b^2}{D},$$

(4) $$\frac{dP}{db} = -Pb\frac{A^2+a^2+b^2}{D}+QA\frac{2ab}{D} = \frac{4abr_1E}{D},$$

(5) $$a\frac{dP}{da} = -A\frac{dP}{dA}-b\frac{dP}{db} = P+a\frac{d\Omega}{db},$$

(6) $$\frac{dQA}{dA} = -A\frac{d\Omega}{db} = -PaA\frac{A^2-a^2-b^2}{D}+QA^2\frac{A^2-a^2+b^2}{D},$$

(7) $$\frac{dQA}{db} = A\frac{d\Omega}{dA} = -PaA\frac{2Ab}{D}+QAb\frac{A^2+a^2+b^2}{D},$$

with the check formulas

(8) $$a\frac{dP}{dA}-A\frac{dQ}{dA}-b\frac{d\Omega}{dA} = 0,$$

$$a\frac{dP}{db}-A\frac{dQ}{db}-b\frac{d\Omega}{db} = 0,$$

$$a\frac{dP}{da}-A\frac{dQ}{da}-b\frac{d\Omega}{da} = 0,$$

$$Pa-QA-\Omega b = W,$$

(9) $$\frac{dW}{dA} = -Q, \qquad \frac{dW}{db} = -\Omega, \qquad \frac{dW}{da} = P.$$

Reviewing these calculations it will be noticed that the S.F. again shows generally a marked superiority over the P.F. in its analytical simplicity.

This N in (1), § 18, is the expression which gives the potential energy (P.E.) of the two co-axial helical currents, or their equivalent current sheet solenoids, investigated by V. JONES, 'Roy. Soc. Proc.,' 1897, or the mutual P.E. of the two pairs of equivalent end plates ('A.M.S.,' 1907, § 59).

But as it is the force only between the two currents which is required, and this is given by $dN/db = L$, we need only calculate the end values of L for measurement in the current weigher.

20. As another illustration of the extra analytical simplicity of the S.F., take the calculations of BROMWICH ('L.M.S.,' 1912), where the results are expressed in a series for the attraction and P.F. of a circular disc, the circle on AB, where the surface density is $\sigma = ky^n$, varying as the $n^{\text{th}}$ power of the distance $y$ from the centre O.

The ring P.F. of a circular element is

(1) $$dP = \int \frac{\sigma\, dy\,.\, y\, d\theta}{PQ'}, \qquad PQ'^2 = A^2 + 2Ay\cos\theta + y^2 + b^2,$$

and the S.F. is

(2) $$dR = 2\pi\sigma\, dy\,[y\Omega\,(MQ') - P'b].$$

Changing from $\Omega\,(MQ')$ to the form

(3) $$\Omega\,(PZ) = \int \frac{Aa\cos\theta + A^2 + b^2}{PZ^2}\cdot\frac{b\, d\theta}{PQ},$$

$PZ^2 = A^2\sin^2\theta + b^2$, as not involving $a$ or $y$, PZ the perpendicular on OQ, this form of $\Omega\,(PZ)$ is obtained by the dissection of the circle into the sector elements $\frac{1}{2}a^2\, d\theta$ ('A.M.S.,' p. 506, § 48),

(4) $$\begin{aligned} R &= \int 2\pi\sigma y\, dy \int \frac{Ay\cos\theta + A^2 + b^2}{A^2\sin^2\theta + b^2}\cdot\frac{b\, d\theta}{PQ} - \int 2\pi\sigma b\, dy \int \frac{y\, d\theta}{PQ'} \\ &= \iint 2\pi\sigma y\, dy\, \frac{A^2\cos^2\theta}{A^2\sin^2\theta + b^2}\cdot\frac{b\, d\theta}{PQ'} + \iint 2\pi\sigma y\, dy\, \frac{Ay\cos\theta}{A^2\sin^2\theta + b^2}\cdot\frac{b\, d\theta}{PQ'} \\ &= \iint 2\pi\sigma y\, dy\, \frac{A^2\cos^2\theta + Ay\cos\theta}{A^2\sin^2\theta + b^2}\cdot\frac{b\, d\theta}{PQ'}, \end{aligned}$$

or with $\sigma = ky^n$,

(5) $$R = \int \frac{ky^{n+1}\, dy}{PQ'} \int \frac{2\pi A^2\cos^2\theta}{A^2\sin^2\theta + b^2}\,.\, b\, d\theta + \int \frac{ky^{n+2}\, dy}{PQ'} \int \frac{2\pi A\cos\theta}{A^2\sin^2\theta + b^2}\,.\, b\, d\theta.$$

Here the $y$ integrations are effected by the formula of reduction obtained from the integration of

(6) $$\frac{d}{dy}\,(y^{n+1}PQ') = [(n+2)\, y^{n+2} + (2n+3)\, Ay^{n+1}\cos\theta + (n+1)\,(A^2+b^2)\, y^n]\,\frac{1}{PQ},$$

and so R can be obtained in finite terms.

But if we attempt the determination of the P.F. the intractable I puts in an appearance when $n$ is odd.

Consider, for example, the flat lens of § 16, 'A.J.M.,' 1919, where $\sigma = k\left(1 - \frac{y^2}{a^2}\right)$; or for $\sigma = k\left(1 - \frac{y^2}{a^2}\right)^{-\frac{1}{2}}$, as in the distribution of electricity in the circular disc.

21. Taking the form in (3), § 20, it can be resolved into

(1) $$\Omega\,(PZ) = -\Omega_1\,(PZ) + \Omega_2\,(PZ),$$

(2) $$\Omega_1 = \tfrac{1}{2}\int \frac{a - \sqrt{(A^2+b^2)}}{\sqrt{(A^2+b^2)} + A\cos\theta}\cdot\frac{b\, d\theta}{PQ}, \qquad \Omega_2 = \tfrac{1}{2}\int \frac{a + \sqrt{(A^2+b^2)}}{\sqrt{(A^2+b^2)} - A\cos\theta}\cdot\frac{b\, d\theta}{PQ},$$

two III. E.I.'s in the form of B in (10), (11), § 4.

To reduce $\Omega_1$ to this standard form, put

(3) $$2a\surd(\mathrm{A}^2+b^2)+2\mathrm{A}a\cos\theta = m^2(\tau_1-t),$$

$$2a\surd(\mathrm{A}^2+b^2)+2\mathrm{A}a = m^2(\tau_1-t_3), \qquad 2a\surd(\mathrm{A}^2+b^2)-2\mathrm{A}a = m^2(\tau_1-t_2),$$

$$a^2+\mathrm{A}^2+b^2-2a\surd(\mathrm{A}^2+b^2) = [a-\surd(\mathrm{A}^2+b^2)]^2 = m^2(t_1-\tau_1),$$

(4) $$\Omega_1 = \int_{t_2}^{t_3}\frac{\surd-\mathrm{U}_1}{\tau_1-t}\cdot\frac{dt}{\surd\mathrm{T}};$$

and to reduce $\Omega_2$, put

(5) $$2a\surd(\mathrm{A}^2+b^2)-2\mathrm{A}a\cos\theta = m^2(t-\tau_2)$$

$$2a\surd(\mathrm{A}^2+b^2)-2\mathrm{A}a = m^2(t_3-\tau_2), \qquad 2a\surd(\mathrm{A}^2+b^2)+2\mathrm{A}a = m^2(t_2-\tau_2)$$

$$a^2+\mathrm{A}^2+b^2+2a\surd(\mathrm{A}^2+b^2) = [a+\surd(\mathrm{A}^2+b^2)]^2 = m^2(t_1-\tau_2),$$

(6) $$\Omega_2 = \int\frac{\surd-\mathrm{U}_2}{t-\tau_2}\cdot\frac{dt}{\surd\mathrm{T}}.$$

Then the sequence runs

(7) $$\infty > t_1 > \tau_1 > t_2 > t > t_3 > \tau_2 > -\infty,$$

and we take

(8) $$f_1\mathrm{G}' = \int_{\tau_1}^{t_1}\frac{\surd(t_1-t_3)\,d\tau}{\surd-\mathrm{U}} = \mathrm{sn}^{-1}\sqrt{\frac{t_1-\tau_1}{t_1-t_2}} = \mathrm{cn}^{-1}\sqrt{\frac{\tau_1-t_2}{t_1-t_2}} = \mathrm{dn}^{-1}\sqrt{\frac{\tau_1-t_3}{t_1-t_3}},$$

(9) $$f_2\mathrm{G}' = \int_{-\infty}^{\tau_2}\frac{\surd(t_1-t_3)\,d\tau}{\surd-\mathrm{U}} = \mathrm{sn}^{-1}\sqrt{\frac{t_1-t_3}{t_1-\tau_2}} = \mathrm{cn}^{-1}\sqrt{\frac{t_3-\tau_2}{t_1-\tau_2}} = \mathrm{dn}^{-1}\sqrt{\frac{t_2-\tau_2}{t_1-\tau_2}},$$

(10) $$\Omega_1 = \pi f_1+2\mathrm{G}\,\mathrm{zn}\,f_1\mathrm{G}', \qquad \Omega_2 = \pi f_1+2\mathrm{G}\,\mathrm{zs}\,f_2\mathrm{G}',$$

and with $f_2-f_1 = 2f$,

(11) $$\Omega(\mathrm{PZ}) = 2\pi f+2\mathrm{G}\,\mathrm{zn}\,2f\mathrm{G}'+2\mathrm{G}\gamma'\,\mathrm{sn}\,2f\mathrm{G}'$$

$$= \Omega(\mathrm{MQ}) = 2\pi-\Omega.$$

Interpreted geometrically on fig. 2, with

(12) $$\mathrm{sn}\,f_1\mathrm{G}' = \sqrt{\frac{t_1-\tau_1}{t_1-t_2}} = \frac{a-\surd(\mathrm{A}^2+b^2)}{r_2},$$

(13) $$\mathrm{sn}\,(1-f_1)\,\mathrm{G}' = \sqrt{\frac{t_1-t_3}{t_1-t_2}}\sqrt{\frac{\tau_1-t_2}{\tau_1-t_3}} = \frac{r_3}{r_2}\sqrt{\frac{\surd(\mathrm{A}^2+b^2)-\mathrm{A}}{\surd(\mathrm{A}^2+b^2)+\mathrm{A}}},$$

(14) $$\mathrm{sn}\,f_2\mathrm{G}' = \sqrt{\frac{t_1-t_3}{t_1-\tau_2}} = \frac{r_3}{a-\surd(\mathrm{A}^2+b^2)},$$

(15) $$\mathrm{sn}\,(1-f_2)\,\mathrm{G}' = \sqrt{\frac{t_3-\tau_2}{t_2-\tau_2}} = \sqrt{\frac{\surd(\mathrm{A}^2+b^2)-\mathrm{A}}{\surd(\mathrm{A}^2+b^2)+\mathrm{A}}}$$

$$= \frac{b}{\surd(\mathrm{A}^2+b^2)+\mathrm{A}} = \frac{\surd(\mathrm{A}^2+b^2)-\mathrm{A}}{b} = \gamma'\,\mathrm{sn}\,(1-f_1)\,\mathrm{G}'.$$

We may drop G′ without ambiguity, and then

(16) $$\gamma' \operatorname{sn} f_1 \operatorname{sn} f_2 = \frac{a-\sqrt{(A^2+b^2)}}{a+\sqrt{(A^2+b^2)}},$$

(17) $$\operatorname{cn} f_1 \operatorname{dn} f_1 = \frac{\sqrt{(\tau_1-t_2 \,.\, \tau_1-t_3)}}{t_1-t_2 \,.\, t_1-t_3} = \frac{2ab}{r_2 r_3},$$

(18) $$\operatorname{cn} f_2 \operatorname{dn} f_2 = \frac{\sqrt{(t_3-\tau_2 \,.\, t_2-\tau_2)}}{t_1-\tau_2} = \frac{2ab}{[a+\sqrt{(A^2+b^2)}]^2},$$

(19) $$\operatorname{sn}(f_2-f_1) = \frac{\operatorname{sn} f_2 \operatorname{cn} f_1 \operatorname{dn} f_1 - \operatorname{sn} f_1 \operatorname{cn} f_2 \operatorname{dn} f_2}{1-\gamma'^2 \operatorname{sn}^2 f_1 \operatorname{sn}^2 f_2} = \frac{b}{r_2} = \operatorname{sn} 2f\mathrm{G}'$$

$$f_2-f_1 = 2f, \qquad \operatorname{cn} 2f\mathrm{G}' = \frac{a-A}{r_2}, \qquad \operatorname{dn} 2f\mathrm{G}' = \frac{a+A}{r_2}.$$

(20) $$\operatorname{sn}(f_2+f_1) = \frac{ab}{r_2\sqrt{(A^2+b^2)}} = \frac{a}{\sqrt{(A^2+b^2)}} \sin \mathrm{BOP} = \frac{\mathrm{OB}}{\mathrm{OP}} \sin \mathrm{BOP}$$

$$= \frac{\mathrm{OP}'}{\mathrm{OB}} \sin \mathrm{BOP} = \sin \mathrm{OBP}',$$

$\mathrm{OBP}' = \operatorname{am}(f_2+f_1)\,\mathrm{G}' = \operatorname{am} 2f'\mathrm{G}'$, suppose.

When OP is produced to cut the circle on AB in R, and the circle on ED again in P′, PP′ will touch the co-axial circle in R; and by the poristic property of these circles with the elliptic function interpretation,

(21) $$\mathrm{OBP}_1 = \operatorname{am} f_1\mathrm{G}', \qquad \mathrm{OBP}_2 = \operatorname{am} f_2\mathrm{G}',$$

if the tangent at the lowest point $e$, $e'$ of the R circle, and of the other co-axial circle touching P′P″, where EP″ = EP, cuts the circle on ED in $P_1$ and $P_2$.

22. Treating Ω (PR) of (3), § 9, in the same way

(1) $$\Omega(\mathrm{PR}) = \Omega_3+\Omega_4-\int \frac{b\,d\theta}{\mathrm{PQ}},$$

(2) $$\Omega_3 = \tfrac{1}{2}\int \frac{\sqrt{(a^2+b^2)}-A}{\sqrt{(a^2+b^2)}+a\cos\theta}\,\frac{b\,d\theta}{\mathrm{PQ}}, \qquad \Omega_4 = \tfrac{1}{2}\int \frac{\sqrt{(a^2+b^2)}+A}{\sqrt{(a^2+b^2)}-a\cos\theta}\,\frac{b\,d\theta}{\mathrm{PQ}},$$

and a similar reduction will give

(3) $$\Omega_3 = \int \frac{\sqrt{-U_3}}{\tau_3-t} \cdot \frac{dt}{\sqrt{T}} = \pi f_3 + 2\mathrm{G} \operatorname{zn} f_3\mathrm{G}',$$

(4) $$\Omega_4 = \int \frac{\sqrt{-U_4}}{t-\tau_4} \cdot \frac{dt}{\sqrt{T}} = \pi f_4 + 2\mathrm{G} \operatorname{zs} f_4\mathrm{G}',$$

(5) $$\infty > t_1 > \tau_3 > t_2 > t > t_3 > \tau_4 > -\infty,$$

(6) $$\operatorname{sn} f_3 G' = \frac{\sqrt{(a^2+b^2)}-A}{r_2}, \qquad \operatorname{sn} f_4 G' = \frac{r_3}{\sqrt{(a^2+b^2)}+A},$$

(7) $$\operatorname{sn}(1-f_4)G' = \sqrt{\frac{\sqrt{(a^2+b^2)}-a}{\sqrt{(a^2+b^2)}+a}} = \frac{b}{\sqrt{(a^2+b^2)}+a} = \gamma' \operatorname{sn}(1-f_3)G',$$

and so on, with $f_3+f_4 = 2f$. Because

(8) $$\gamma' \operatorname{sn} f_3 \operatorname{sn} f_4 = \frac{\sqrt{(a^3+b^2)}-A}{\sqrt{(a^2+b^2)}+A},$$

(9) $$\operatorname{cn} f_3 \operatorname{dn} f_3 = \frac{2Ab}{r_2 r_3}, \qquad \operatorname{cn} f_4 \operatorname{dn} f_4 = \frac{2Ab}{[\sqrt{(a^2+b^2)}+A]^2},$$

(10) $$\operatorname{sn}(f_4+f_3) = \frac{b}{r_2} = \operatorname{sn} 2fG',$$

(11) $$\operatorname{sn}(f_4-f_3) = \frac{Ab}{r_2\sqrt{(a^2+b^2)}} = \frac{A}{\sqrt{(a^2+b^2)}} \sin OBP$$

Produce NP parallel to AB to cut the circle on ED again in $P_4$, then

(12) $$\frac{A}{\sqrt{(a^2+b^2)}} = \frac{NP}{NB} = \frac{BP}{BP_4},$$

because P, $P_4$ are inverse points in the circle, centre N, through B; so that

(13) $$\operatorname{sn}(f_4-f_3) = \frac{BP}{BP_4} \sin BPP_4 = \sin BP_4P,$$

(14) $$BP_4P = \operatorname{am}(f_4-f_3)G', \qquad OBP_4 = \operatorname{am}(2-f_4+f_3)G'.$$

Thence a geometrical construction for $\operatorname{am} f_3 G'$ and $\operatorname{am} f_4 G'$, similar to that above for $f_1$ and $f_2$.

The pole of the chord RR′ through B will lie on the line through A perpendicular to AB, at A′ suppose, and the tangent A′R′ will be parallel to AB.

A whole chapter might be written of elliptic function theory, showing in this manner the geometrical interpretation of the various formulas, especially of the quadric transformation, in relation to co-axial circles.

23. Our chief object was to employ a straightforward integration of MAXWELL'S result as a direct road to the analytical results required in ampère-balance current weighing. The check on the arithmetical calculations has been explained and carried out in the 'Transactions of the American Mathematical Society' ('A.M.S.'), 1907, § 56, p. 516.

Considering that the chief analytical and numerical difficulties in these operations arise in the III. E.I. expression of Ω, and that this is cancelled by making

(1) $$A = a, \qquad f = \tfrac{1}{2}, \qquad \Omega = \pi - 2G\gamma' = \pi - 2K(1-\kappa),$$

$\gamma' = \cos$ APB, so that APB is the modular angle,

$$(2) \qquad L = \pi Pab + \tfrac{1}{2}Mb, \qquad N = \pi Pa\left(\tfrac{4}{3}a^2 + b^2\right) + M\left(\tfrac{2}{3}a^2 - \tfrac{1}{6}b^2\right),$$

involving only the complete E.I. I. and II., given in LEGENDRE's tables with extreme accuracy, it would appear to be of practical advantage to make all the helical coils of the same diameter.

This would prevent one coil from going inside another, and they would require to be opposed in axial prolongation, as in the Lorenz apparatus at Teddington, described in 'Phil. Trans.,' 1913, by F. E. SMITH.

Here is a question to be decided by practical experience as to the advantage or defects of this suggestion.

The current weigher is designed for legal commercial use, in the definition of the electrical units in an Act of Parliament, and these require to be measured to as many significant figures as possible, warranted by the most careful measurement of skilled observers.

The legal definition must be specified with the same precision of language as we find in the Act of Parliament on Weights and Measures, defining the standard pound and yard, the length of the seconds pendulum with a view of checking and preserving the standard, the volume of the gallon in cubic inches, and other standards of measure in civilised life.

## LIST OF REFERENCES.

VIRIAMU JONES, 'Phil. Mag.,' 1889; 'Phil. Trans.,' 1891; 'Roy. Soc. Proc.,' 1897.
G. M. MINCHIN, 'Phil. Mag.,' 1893–4.
W. BURNSIDE, 'Mess. of Math.,' 1891.
COLERIDGE FARR, 'Roy. Soc. Proc.,' 1898.
COFFIN, ROSA, COHEN, 'Bulletin of the Bureau of Standards,' 1906.
NAGAOKA, 'Journal of the Tokyo College of Science,' 1903; 'Phil. Mag.,' 1918.
A. RUSSELL, 'Phil. Mag.,' 1907.
F. E. SMITH, 'Phil. Trans.,' A, vol. 214, 1914.
S. BUTTERWORTH, 'Phil. Mag.,' 1915.